One-ended spanning subforests in pmp graphs of superquadratic growth

Last time ...

- The number of *ends* of G is the supremum of the number of infinite connected components in $G|(X \setminus F)$ for $F \subseteq G$ finite.

Theorem

If G is hyperfinite, pmp, and almost nowhere 0 or 2 ended then G has an a.e. one-ended treeing.

This time ...

Theorem (2.6) Suppose G is a pmp locally finite Borel graph on (X,μ) of superquadratic growth (i.e., $\exists c > 0$ s.t. $\forall^{\mu} x \ \forall r \ |B_r(x)| \geq c r^2$) then G has a Borel a.e. one-ended spanning subforest.

Note: Isoperimetric constant φ of G is

$$\varphi := \inf_{\substack{A \subseteq X \\ \mu(A) > 0 \\ G|_A \text{ is component finite}}} \frac{\mu(\partial_G A)}{\mu(A)} \qquad \{x \notin A : \exists a \in A \ x \, G \, a\}$$

If $\varphi > 0$ then G has exponential growth, so Theorem applies.

Lemma (2.5) G a loc. finite pmp Borel graph on (X, μ) & there are partial Borel functions $f_0, f_1, \ldots \subseteq G$ s.t.

(1) $\bigcup \mathrm{dom}(f_i) = X$

(2) $\sum \mu(\mathrm{dom}(f_i)) < \infty$

(3) Each f_i is aperiodic & has finite back orbits, i.e., $\forall x$ $\{y \in X : \exists n\, f^n y = x\}$ finite

(4) $\forall i$ & $\forall x \in \mathrm{dom}(f_i)$ there is $j \geq i$ with $f_i(x) \in \mathrm{dom}(f_j)$

$$x \xrightarrow{i} f_i(x) \overset{j \geq i}{\dashrightarrow} f_j(f_i(x))$$

Then G has a Borel a.e. one-ended spanning subforest

Idea: Put the f_is together to get a Borel (full) $f : X \to X \subseteq G$ aperiodic & has finite back orbits

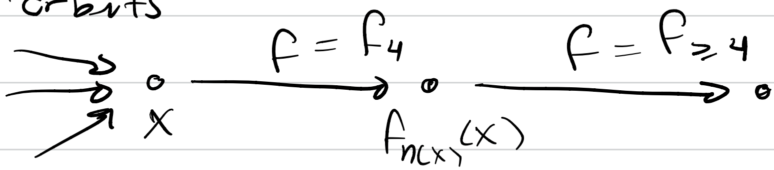
Proof $B := \{x : \exists^\infty i \;\; x \in \mathrm{dom}(f_i)\}$ is null

$= \limsup (\mathrm{dom}(f_i))$

So we may assume WLOG that $\forall x \in X$
$\{i \in \mathbb{N} : x \in \mathrm{dom}(f_i)\}$ is nonempty & finite

Define
$$f : X \to X \subseteq G$$
$$x \mapsto f_{n(x)}(x)$$
where $n(x) := \max\{i \in \mathbb{N} : x \in \mathrm{dom}(f_i)\}$

By (4), $n(x)$ is non-decreasing along f-orbits



So f is aperiodic by this fact & (3)

By (3), each f_i has finite back orbits so f does as well

$\square$

Theorem (2.6) Suppose G is a pmp locally finite Borel graph on (X,μ) of superquadratic growth
(i.e., $\exists c > 0$ s.t. $\forall^\mu x\ \forall r\ |B_r(x)| \geq cr^2$)
then G has a Borel a.e. one-ended spanning subforest

Proof: We'll come up with $f_i \subseteq G$ satisfying (1) – (4) from lemma

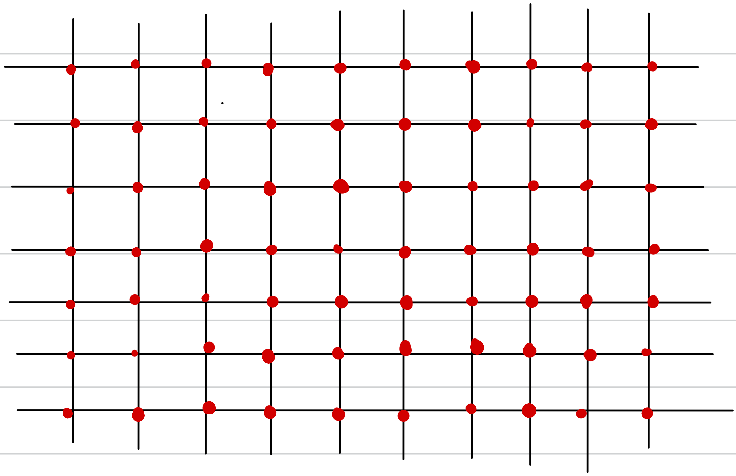
Let $r_n := 2^n$
(so $\sum_n \frac{2r_{n+1}}{cr_n^2} < \infty$)

Put $A_0 := X$

For $n \geq 1$ let A_n be Borel & maximal wrt the fact that for $x \neq y \in A_n$,

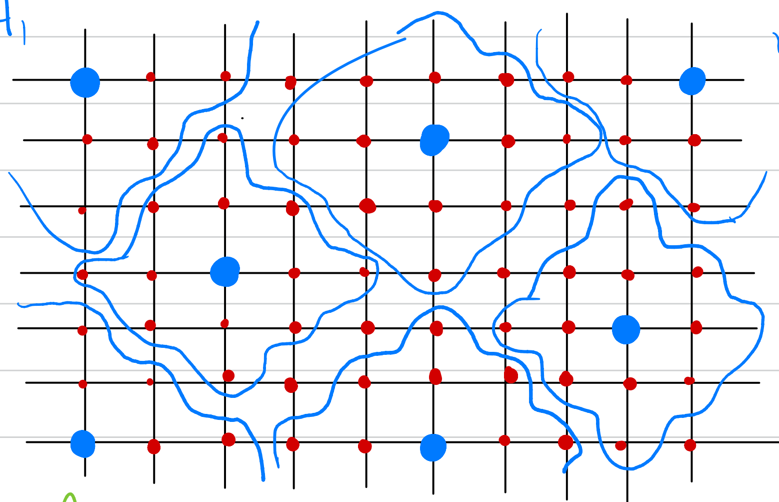
$$B_{r_n}(x) \cap B_{r_n}(y) = \emptyset$$

A_0



A_1

$r_1 = 2$

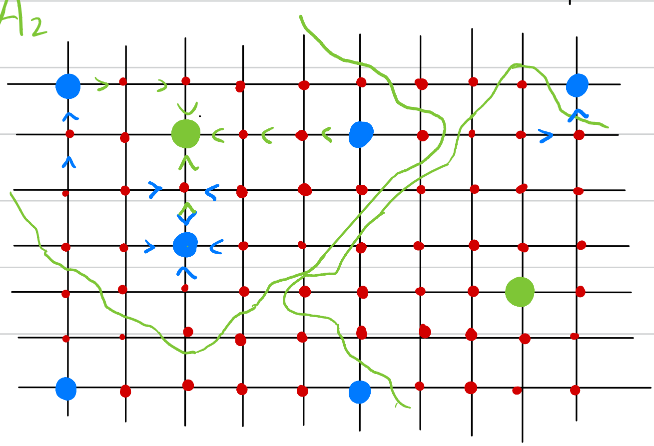


A_2

$r_2 = 4$

f_0

f_1



Def. of f_n : for $x \in A_n$ take lex. least minimal length path from x to a point $y \in A_{n+1}$

$$x = x_0, x_1, x_2, \ldots\ldots, x_k = y$$

Define f_n to be the union of the pairs (x_i, x_{i+1}) in these paths

So $f_i \subseteq G_J$ are partial Borel functions.

(1) holds since $A_n \subseteq \mathrm{dom}(f_n)$

$$A_0 = X$$

(3) : aperiodic & finite back orbits

minimality of paths & lengths of paths are bounded (by maximality of A_ns)

(4) : $x \in \mathrm{dom}(f_i) \Rightarrow f_i(x) \in \mathrm{dom}(f_j)$
$j \geq i$

If $f_i(x)$ was in the middle of a path then $f_i(x) \in \mathrm{dom}(f_i)$

Otherwise $f_i(x) \in A_{i+1} \subseteq \mathrm{dom}(f_{i+1})$

(2) : $\sum \mu(\mathrm{dom}(f_i)) < \infty$

By maximality of A_{n+1}, the length of a path from $x \in X$ to $y \in A_{n+1}$ $\leq 2r_{n+1}$

$$\Rightarrow \mu(\mathrm{dom}(f_n)) \leq 2r_{n+1}\, \mu(A_n) \leq \frac{2r_{n+1}}{cr_n^2}$$

since $\mu(A_n) \leq \frac{1}{cr_n^2}$

Define $E_n :=$ being in the same ball of radius r_n for $x \in A_n$

$$\int f(x)\, d\mu(x) = \int \text{Average of } f \text{ over equivalence class of } x \; d\mu(x)$$

$\square$

Isoperimetric constant $> \varphi > 0$
then we have exponential growth
$\forall^{\mu} x \in X$
$$|B_n(x)| \geq (1+\varphi)^n$$

Proof by induction $|B_0(x)| = 1$ ✓

Suffices to show $|B_{n+1}(x)| \geq (1+\varphi)|B_n(x)|$
for μ-a.e. $x \in X$

If not, we can take a maximal
disjoint collection of balls of radius $n+1$
(which is Borel) around points x
with $|B_{n+1}(x)| < (1+\varphi)|B_n(x)|$

$$A := \bigcup_{\substack{x \text{ center} \\ \text{of ball in} \\ \text{collection}}} B_n(x)$$

A is Borel,
positively measured,
$G|_A$ is component
finite

$$\mu(\partial_G A) \leq \varphi \mu(A) \Longrightarrow \frac{\mu(\partial_G A)}{\mu(A)} \leq \varphi$$

$\square$